USE OF MONTE CARLO METHOD FOR SOLUTION OF TIME-DEPENDENT PROBLEMS IN THE HYDRODYNAMICS OF VISCOPLASTIC MEDIA

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A scheme is proposed for numerical solution of time-dependent problems of plane flow of viscoplastic media by the Monte Carlo method. Solutions are given for the problems of development and damping of gradient flow and damping of Couette flow.

A viscoplastic medium is a rheologic solid for which the dependence of shear stress on rate of deformation for plane motions has the form

$$\tau = \mu \frac{\partial u}{\partial y} + \tau_0 \operatorname{sign} \frac{\partial u}{\partial y}; \quad |\tau| \geqslant \tau_0, \tag{1}$$

where τ_0 and μ are respectively the limiting shear stress and the coefficient of dynamic viscosity (rheologic constants) of the medium; the coordinate axis is perpendicular to the flow velocity in the medium. When $|\tau| > \tau_0$, a viscoplastic medium "flows" (the tensor for the rate of deformation is different from zero); when $|\tau| < \tau_0$, a viscoplastic medium is an absolutely rigid body.

A property of time-independent and time-dependent flows in a viscoplastic medium in plane channels [1, 2] is the possibility of formation of a zone of quasirigid motion in which $|\tau| < \tau_0$. The solution of time-independent problems of plane flows of viscoplastic media in a channel with parallel walls presents no difficulties. For a limited class of time-dependent problems which have self-similarity, a solution can be obtained in analytic form [3]. Investigation of the problem for the general case is accompanied by considerable mathematical difficulty [4]. The use of numerical methods is probably more effective in obtaining specific results.

We consider time-dependent flow of a viscoplastic medium in a plane channel ($0 \leq y \leq 2L$) under the action of a pressure gradient P(t). The equation of motion takes the form

$$\rho \frac{\partial u}{\partial t} = -P(t) + \frac{\partial \tau}{\partial y}, \tag{2}$$

where ρ is the density of the medium; τ for the zone of viscous flow is defined by Eq. (1); in the zone of quasirigid motion, the velocity of the medium is independent of the coordinate:

$$u = u_0(t). \tag{3}$$

Differentiating Eq. (2) with respect to y and making use of (1), we obtain after transformation to the dimensionless variables $\xi = \mu t/\rho L^2$ and $\eta = y/L$ an equation for the shear stress τ, which is relative to some characteristic shear stress τ_{ch} for the problem,

$$\frac{\partial \tau}{\partial \xi} = \frac{\partial^2 \tau}{\partial \eta^2}. \tag{4}$$

Making use of the symmetry of the problem, we consider flow in the lower half of the channel ($0 \leq \eta \leq 1$). At the fixed wall $\eta = 0$, we have $U = \partial U/\partial t = 0$, and consequently from Eq. (2)

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$$\frac{\partial\tau}{\partial\eta}(\xi,\ 0)=w(\xi);\quad w(\xi)=\frac{PL}{\tau_{ch}}. \tag{5}$$

At the boundary of the quasirigid core $\eta_0(\xi)$, we have from the definition of a viscoplastic medium

$$\tau\left[\xi,\ \eta_0(\xi)\right]=s;\quad s=\frac{\tau_0}{\tau_{ch}}. \tag{6}$$

From Eqs. (2) and (3), it follows that $\partial\tau/\partial y$ is constant over the cross section of the quasirigid zone at each point in time. Using Eq. (6), we have

$$\frac{\partial\tau}{\partial\eta}(\xi,\ \eta)=-\frac{s}{1-\eta_0(\xi)}\qquad(1\geqslant\eta\geqslant\eta_0(\xi)).$$

Considering the continuity of medium velocity and shear stress in the transition across the zone boundary, we arrive at a condition for $\tau(\xi,\ \eta)$ at the boundary of the viscous zone:

$$\frac{\partial\tau}{\partial\eta}\left[\xi,\ \eta_0(\xi)\right]=-\frac{s}{1-\eta_0(\xi)}. \tag{7}$$

Note that in problems with no initial zone of viscous flow, i.e., for $\eta_0(0)=0$, the effect of initial conditions on the solution for the set of various $(\xi,\ \eta)$ is insignificant except for the point (0, 0) [5].

We consider certain types of time-dependent motions in a viscoplastic medium.

1. In a viscoplastic medium in a channel with fixed parallel walls, which is at rest for $\xi<0$, let a time-dependent flow arise through the action of an instantaneously applied pressure gradient that is constant in time. In this case

$$\eta_0(0)=0,\quad(w=-1). \tag{8}$$

For large ξ, the solution $\eta_0(\xi)$ must tend to the value

$$\eta_0(\infty)=1-s, \tag{9}$$

which corresponds to time-independent flow.

2. For $\xi<0$, let a time-independent flow of a viscoplastic medium in a channel with fixed parallel walls be maintained by a pressure gradient $|P|=\tau_{ch}/L$ which is constant in time. At the time $\xi=0$ and subsequently, we make $P=0$. In this case

$$\eta_0(0)=1-s;\quad\tau(0,\ \eta)=1-\eta,\quad(w=0). \tag{10}$$

The motion just described we call damping of gradient flow.

3. For $\xi<0$, let a time-independent flow in a viscoplastic medium in a channel with fixed parallel walls be created by a constant force applied to an infinitely thin, weightless, rigid layer located in the center of the channel and parallel to its walls which moves in its own plane at a constant velocity. At the time $\xi=0$, the central layer is left to its own resources. In this case, the initial conditions are

$$\eta_0(0)=1,\quad\tau(0,\ \eta)=1,\quad(w=0). \tag{11}$$

We call Case 3 the damping of symmetric Couette flow.

The problems described belong to the class of mixed nonlinear problems with unknown boundary for the equation of thermal conductivity. Case 1 was investigated in [6]; however, the solution obtained was incorrect because they used in place of (7) in the system of boundary conditions the condition

$$\tau(\xi,\ 1)=0,$$

which is only valid for the zone of quasirigid motion. It cannot be used as a boundary condition for the viscous zone because the line $(\xi,\ 1)$ lies entirely within the region for which Eq. (4) is not satisfied.

A similar problem for medium velocity $u(\xi,\ \eta)$ was reduced in sufficiently general formulation to a nonlinear system of integral Volterra equations of the first kind with an asymptotic solution of the problem being obtained for small time values in Case 1 [2].

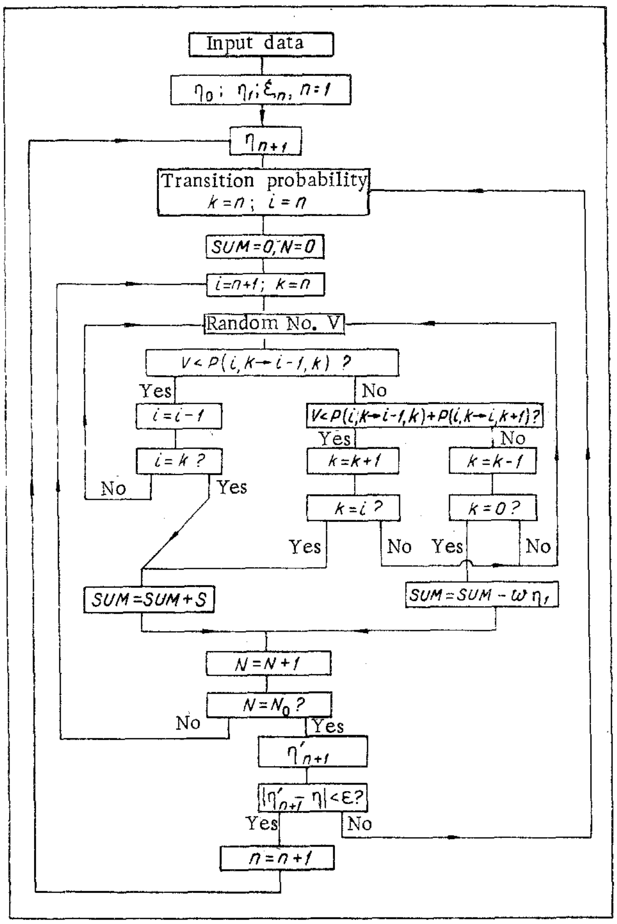


Fig. 1. Block diagram of program for realization of the Monte Carlo method as applied to problem 1. $N_0 = 2000$.

In this paper, the Monte Carlo method is used to obtain solutions suitable for any $\xi > 0$. Application of the method of statistical trials (Monte Carlo) to the solution of boundary value and mixed problems in mathematical physics has been described [7, 8]. Haji-Sheikh and Sparrow [9] successfully applied this method to the solution of a problem with unknown boundary for the equation of thermal conductivity – the Stefan freezing problem. In contrast to the problem discussed in [9], the boundary conditions in the problems being discussed (Cases 1-3) do not contain time derivatives of functions describing the position of the zone boundary, which eliminates the possibility of using the method of solution developed in [9].

Before realization of the Monte Carlo method, the mixed (or boundary value) problem must be converted to finite-difference form and the coefficients of the finite-difference equations must be interpreted as probabilities for transition of the moving point from one point of the difference mesh to another. We present expressions for the transition probabilities in an implicit difference scheme for the equation of thermal conductivity defined over a four-point pattern $(\xi_{i-1};\eta_k)$, $(\xi_i;\eta_{k+1})$, $(\xi_i;\eta_k)$, (ξ_i,η_{k-1}) in a nonuniform grid of ξ and η:

$$P(i,\ k\to i-1,\ k)=\left[1+\frac{\xi_i-\xi_{i-1}}{(\eta_{k+1}-\eta_k)^2}\cdot\frac{\eta_{k+1}-\eta_{k-1}}{\eta_k-\eta_{k-1}}\right]^{-1},$$

$$P(i,\ k\to i,\ k+1)=\left[\frac{(\eta_{k+1}-\eta_k)^2}{\xi_i-\xi_{i-1}}+\frac{\eta_{k+1}-\eta_{k-1}}{\eta_k-\eta_{k-1}}\right]^{-1}, \tag{12}$$

$$P(i,\ k\to i,\ k-1)=\left[\frac{(\eta_{k+1}-\eta_k)(\eta_k-\eta_{k-1})}{\xi_i-\xi_{i-1}}+\frac{\eta_{k+1}-\eta_{k-1}}{\eta_{k+1}-\eta_k}\right]^{-1},$$

where, for example, $P(i,\ k\to i-1,\ k)$ is the probability for the transition of the moving point from (ξ_i, η_k) to $(\xi_{i-1};\eta_k)$.

In finite-difference form, Eqs. (5)-(7) become

$$\tau[\xi_i;\ 0]=\tau[\xi_i;\ \eta_1]-\omega(\xi_i)\,\eta_1, \tag{13}$$

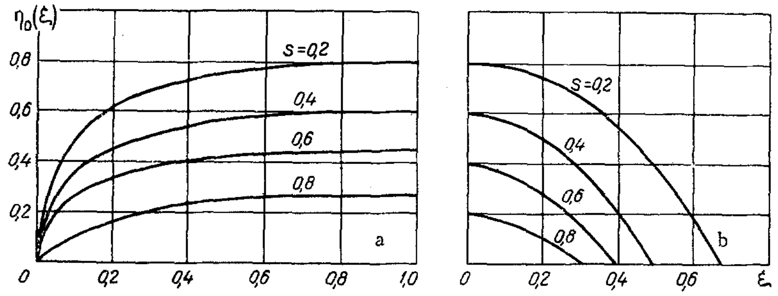
$$\tau[\xi_i;\ \eta_0(\xi_i)]=s, \tag{14}$$

$$\tau[\xi_i;\ \eta_0(\xi_i)]=\tau[\xi_i;\ \eta_0(\xi_{i-1})]-\frac{s\left[\eta_0(\xi_i)-\eta_0(\xi_{i-1})\right]}{1-\eta_0(\xi_i)}, \tag{15}$$

and Eqs. (10) and (11), necessary for solution of the problem in the second and third cases, are respectively written as

$$\tau(0,\ \eta_k)=1-\eta_k, \tag{16}$$

$$\tau(0,\ \eta_k)=1. \tag{17}$$



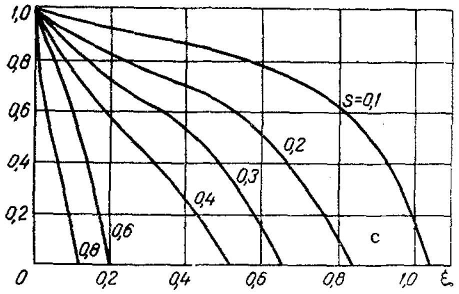


Fig. 2. Time dependence of zone boundary location in the various problems: a) development of gradient flow; b) damping of gradient flow; c) damping of symmetric Couette flow.

The probabilistic significance of Eqs. (13), (14), (16), and (17) is given in [9]. According to the Monte Carlo method, the value of the unknown function $\tau(\xi_i;\ \eta_k)$ is equal to the mathematical expectation value of the free terms in Eqs. (13) and (14), and (16) or (17) respectively in the cases of the second and third equations, which are selected from a general sample for the incidence of moving particles on the boundary or beginning of the region. The central limit theorem of probability theory [8] guarantees, under sufficiently broad assumptions, the probability convergence of the arithmetic mean of the values mentioned to the mathematical expectation value in a finite number of trials.

The algorithm for solving the problem consists of the following. We assume that certain values $\eta_0(\xi_i)$ are known. One can then use as a trial value of $\eta_0(\xi_{i+1})$

$$\eta_0(\xi_{i+1}) = \eta_0(\xi_i) + \frac{\eta_0(\xi_i) - \eta_0(\xi_{i-1})}{\xi_i - \xi_{i-1}}(\xi_{i+1} - \xi_i) \tag{18}$$

and build up the grid to the time ξ_{i+1}. Determining the corresponding transition probabilities (12) and developing the random movement of a finite number of particles, we calculate $\tau[\xi_{i+1};\ \eta_0(\xi_i)]$. Substituting the result in (15), written for the time ξ_{i+1}, and using (14), we obtain a refined value for the quantity $\eta_0(\xi_{i+1})$. If the difference between the assigned and calculated values of $\eta_0(\xi_{i+1})$ is small, we go to the next step; if it is not, the calculated value is used in a repetition of the calculation.

Equation (18) does not permit definition of the value of the first step. For Case 1, this can be done with the help of an asymptotic solution [2]; for the second and third cases, it is convenient to take $\eta_0(\xi_1) = \eta_0(0)$.

A block diagram for realization of the Monte Carlo method as applied to the first problem is shown in Fig. 1. The relationships $\eta_0(\xi)$ for the first, second, and third problems are shown respectively in Fig. 2a, 2b, and 2c for several values of the plasticity parameter. A feature of the solutions for the second and third problems is the finite flow time. Where damping of the flow of a Newtonian fluid under similar conditions is of an asymptotic nature, flow of a viscoplastic material ceases at the moment the boundary of the quasirigid core reaches the channel wall. It is easy to see in Fig. 2a that in the first problem, the position of the zone boundary $\eta_0(\xi)$ tends to the time-independent value in accordance with (9) for sufficiently large ξ. Escape into the time-independent mode is more acceptable for small values of s and deteriorates as the plasticity parameter increases while maintaining a qualitatively correct nature. This phenomenon is intrinsically characteristic of the method of solution because it can be shown that the relative error in determination of $\eta_0(\xi_{i+1})$ increases as s increases.

In the application of the Monte Carlo method to the solution of boundary-value and mixed problems in mathematical physics, one must keep in mind the approximation errors associated with the introduction of a finite-difference scheme and the errors associated with a finite sample. Errors of the first kind can be reduced by using an absolutely stable difference scheme and varying the grid parameters; errors of the second kind are reduced by increasing the number of trials. Both require an increase in machine time for problem solution. In this connection, it is appropriate to point out that the quality of the method is determined not only by the number of arithmetical operations required for its realization but also by the time an investigator spends in preparing for solution of the problem on a computer [10].

NOTATION

τ	is the shear stress;
μ	is the dynamic viscosity;
τ_0	is the limit shear stress;
u	is the velocity of fluid;
t	is the time;
y	is the transverse coordinate;
η	is the dimensionless transverse coordinate;
ξ	is the dimensionless time;
η_0	is the unknown interface;
P	is the pressure gradient;
s	is the plasticity parameter;
$P(i, k \rightarrow i-1, k)$	is the possibility of transfer of flowing point from (ξ_i, η_k) to $(\xi_{i-1}; \eta_k)$.

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